Frobenius P-categories via the Alperin condition

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1. Introduction

1.1. Let P be a finite p-group and $\mathcal{F}$ a *divisible P-category*. In [5, Ch. 5] we showed that our approach in [4, Appendix] to Alperin's Fusion Theorem [1] for *local pointed groups* can be translated to $\mathcal{F}$ and that, in this context, it still makes sense to define the *$\mathcal{F}$-essential* subgroups of P [5, 5.7]. Then, we rose the following question: *in what extend the behaviour of the $\mathcal{F}$-essential subgroups of P characterizes the Frobenius P-categories?* In this Note we give a more satisfactory answer to this question than what we obtained in [5, Theorem 5.22].

1.2. Let us recall our setting. A *divisible P-category* is a subcategory $\mathcal{F}$ of the category of finite groups containing the *Frobenius category* $\mathcal{F}_P$ of P [5, 1.8] — where the objects are all the subgroups of P and where all the homomorphisms are injective — and fulfilling the following condition:

1.2.1. *If Q, R and T are subgroups of P, for any $\varphi \in \mathcal{F}(Q,R)$ and any group homomorphism $\psi : T \to R$ the composition $\varphi \circ \psi$ belongs to $\mathcal{F}(Q,T)$ (if and) only if $\psi \in \mathcal{F}(R,T)$.*

Here, $\mathcal{F}(Q,R)$ denotes the set of $\mathcal{F}$-morphisms from R to Q. Moreover, we consider the category $\mathbb{Z}\mathcal{F}$ still defined over the set of all the subgroups of P where, for any pair of subgroups Q and R of P, the set of morphisms from R to Q is the *free $\mathbb{Z}$-module* $\mathbb{Z}\mathcal{F}(Q,R)$ over $\mathcal{F}(Q,R)$, with the *distributive* composition extending the composition in $\mathcal{F}$.

1.3. For any two different elements $\varphi, \varphi' \in \mathcal{F}(Q,R)$, we call *$\mathcal{F}$-dimorphism* from R to Q the difference $\varphi' - \varphi$ in the $\mathbb{Z}$-module $\mathbb{Z}\mathcal{F}(Q,R)$; it is clear that the set of $\mathcal{F}$-dimorphisms is stable by left-hand and right-hand composition with $\mathcal{F}$-morphisms; note that, for any $\varphi \in \mathcal{F}(Q,R)$, the family $\{\varphi' - \varphi\}_{\varphi'}$, where φ' runs over $\mathcal{F}(Q,R) - \{\varphi\}$, is a $\mathbb{Z}$-basis of the kernel of the evident *augmentation* $\mathbb{Z}$-linear map

$$\varepsilon_{Q,R} : \mathbb{Z}\mathcal{F}(Q,R) \longrightarrow \mathbb{Z} \qquad 1.3.1$$

sending any $\varphi \in \mathcal{F}(Q,R)$ to 1.

1.4. The next elementary lemma [5, Lemma 5.4] relates any "linear" decomposition of an $\mathcal{F}$-dimorphism in terms of $\mathcal{F}$-dimorphisms with the old *partially defined linear combinations* introduced in [3, Ch. III]. Note that, in

the case where $Q = P$, φ is the inclusion map $\iota_R^P : R \to P$, and for any $i \in I$, we have $\mu_i = \iota_{Q_i}^P$, $Q_i = R_i$ and $\varphi_i = \mathrm{id}_{R_i}$, equalities 1.5.2 below coincide with the decomposition pattern in the original formulation of Alperin's Fusion Theorem [1].

Lemma 1.5. *With the notation above, let $\{Q_i\}_{i\in I}$ and $\{R_i\}_{i\in I}$ be finite families of subgroups of P and, for any $i \in I$, let $\varphi'_i - \varphi_i$ be an $\mathcal{F}$-dimorphism from R_i to Q_i and $\mu_i : Q_i \to Q$ and $\nu_i : R \to R_i$ be two $\mathcal{F}$-morphisms. Then, we have*

$$\varphi' - \varphi = \sum_{i\in I} \mu_i \circ (\varphi'_i - \varphi_i) \circ \nu_i \qquad 1.5.1$$

if and only if there are $n \in \mathbb{N}$ and an injective map $\sigma : \Delta_n \to I$ fulfilling

$$\begin{gathered} \varphi = \mu_{\sigma(0)} \circ \varphi_{\sigma(0)} \circ \nu_{\sigma(0)} \\ \mu_{\sigma(\ell-1)} \circ \varphi'_{\sigma(\ell-1)} \circ \nu_{\sigma(\ell-1)} = \mu_{\sigma(\ell)} \circ \varphi_{\sigma(\ell)} \circ \nu_{\sigma(\ell)} \ \textit{ for any } 1 \le \ell \le n \qquad 1.5.2 \\ \mu_{\sigma(n)} \circ \varphi'_{\sigma(n)} \circ \nu_{\sigma(n)} = \varphi' . \end{gathered}$$

1.6. According to *Yoneda's Lemma* [2, §1], the *contravariant* functor $\mathfrak{h}_{\mathcal{F}} : \mathcal{F} \to \mathfrak{Ab}$ mapping any subgroup Q of P on $\mathbb{Z}\mathcal{F}(P,Q)$ and any $\mathcal{F}$-morphism $\varphi : R \to Q$ on the group homomorphism $\mathfrak{h}_{\mathcal{F}}(Q) \to \mathfrak{h}_{\mathcal{F}}(R)$ defined by the composition with φ is a *projective object* in the category of *contravariant* functors from $\mathcal{F}$ to $\mathfrak{Ab}$. Then, denoting by $\mathbb{Z} : \mathcal{F} \to \mathfrak{Ab}$ the trivial *contravariant* functor mapping any $\mathcal{F}$-object on $\mathbb{Z}$, the ring of integers, and any $\mathcal{F}$-morphism on $\mathrm{id}_{\mathbb{Z}}$, the family of *augmentation maps* $\varepsilon_{P,Q}$ when Q runs over the set of subgroups of P defines a surjective *natural map*

$$\varepsilon_{\mathcal{F}} : \mathfrak{h}_{\mathcal{F}} \longrightarrow \mathbb{Z} \qquad 1.6.1$$

and we set $\mathfrak{w}_{\mathcal{F}} = \mathrm{Ker}(\varepsilon_{\mathcal{F}})$, which is nothing but the *Heller translated* of the trivial functor $\mathbb{Z}$.

1.7. On the other hand, if $\mathfrak{a} : \mathcal{F} \to \mathfrak{Ab}$ is a *contravariant* functor, let us say that a family $\mathcal{S} = \{S_Q\}_Q$ of subsets $S_Q \subset \mathfrak{a}(Q)$, where Q runs over the set of proper subgroups of P, is a *generator family of* $\mathfrak{a}$ whenever, for any proper subgroup Q of P, we have

$$\mathfrak{a}(Q) = \sum_R \sum_{\varphi\in\mathcal{F}(R,Q)} \sum_{a\in S_R} \mathbb{Z}\cdot\big(\mathfrak{a}(\varphi)\big)(a) \qquad 1.7.1,$$

where R runs over the set of subgroups of P (such that $|R| \ge |Q|$). Now, it is quite clear from Lemma 1.5 above that Alperin's Fusion Theorem [1] provides a particular *generator family* of the *Heller translated* $\mathfrak{w}_{\mathcal{F}}$.

1.8. In order to find minimal *generator families* of $\mathfrak{w}_{\mathcal{F}}$, let us define a subfunctor $\mathfrak{r}_{\mathcal{F}}$ of $\mathfrak{w}_{\mathcal{F}}$ mapping any subgroup Q of P on

$$\mathfrak{r}_{\mathcal{F}}(Q) = w_{\mathcal{F}}(P) \circ \mathbb{Z}\mathcal{F}(P,Q) + \sum_R \mathfrak{w}_{\mathcal{F}}(R) \circ \mathbb{Z}\mathcal{F}(R,Q) \qquad 1.8.1$$

where R runs over the set of subgroups of P such that $|R| > |Q|$. Then, we say that Q is $\mathcal{F}$-*essential* whenever $\mathfrak{r}_\mathcal{F}(Q) \neq \mathfrak{w}_\mathcal{F}(Q)$ and call $\mathcal{F}$-*irreducible* the elements of $\mathfrak{w}_\mathcal{F}(Q) - \mathfrak{r}_\mathcal{F}(Q)$. Coherently, the elements of $\mathfrak{r}_\mathcal{F}(Q)$ are called $\mathcal{F}$-*reducible*; actually, any element of $\mathfrak{r}_\mathcal{F}(Q)$ is a sum of a family of $\mathcal{F}$-reducible $\mathcal{F}$-*dimorphisms* from Q to P . The following result [5, Proposition 5.9] justifies all these definitions.

Proposition 1.9. *Let $\mathcal{S} = \{S_Q\}_Q$ be a generator family of $\mathfrak{w}_\mathcal{F}$, where Q runs over the set of proper subgroups of P . The family formed by the $\mathcal{F}$-irreducible elements of S_Q , where Q runs over the set of $\mathcal{F}$-essential subgroups of P , is also a generator family of $\mathfrak{w}_\mathcal{F}$. Moreover, for any $\mathcal{F}$-essential subgroup Q of P , there is $\varphi \in \mathcal{F}(P,Q)$ such that $S_{\varphi(Q)}$ contains an $\mathcal{F}$-irreducible element.*

1.10. Before going further, recall that $\mathcal{F}$ is a *Frobenius P-category* whenever it fulfills the following two conditions [5, 2.8]

Sylow condition. *The group $\mathcal{F}_P(P)$ of inner automorphisms of P is a Sylow p-subgroup of $\mathcal{F}(P)$.*

Extension condition. *For any subgroup Q of P , any subgroup K of* $\mathrm{Aut}(Q)$ *and any $\mathcal{F}$-morphism $\varphi\colon Q \to P$ such that $\varphi(Q)$ is fully ${}^{\varphi}K$-normalized in $\mathcal{F}$, there are an $\mathcal{F}$-morphism $\psi\colon Q{\cdot}N_P^K(Q) \to P$ and $\chi \in K$ such that $\psi(u) = \varphi\big(\chi(u)\big)$ for any $u \in Q$.*

Here we set $\mathcal{F}(Q) = \mathcal{F}(Q,Q)$, ${}^{\varphi}K$ denotes the image of K in $\mathrm{Aut}\big(\varphi(Q)\big)$ throughout the isomorphism $Q \cong \varphi(Q)$ induced by φ and we say that Q *is fully K-normalized in $\mathcal{F}$* whenever it fulfills [5, 2.6]

1.10.1 *For any $\psi \in \mathcal{F}\big(P, Q{\cdot}N_P^K(Q)\big)$, we have $\psi\big(N_P^K(Q)\big) = N_P^{{}^{\psi}K}\big(\psi(Q)\big)$.*

Recall that we say *fully centralized* or *fully normalized* whenever $K = \{1\}$ or $K = \mathrm{Aut}(Q)$, replacing $N_\mathcal{F}^K$ by $C_\mathcal{F}$ or $N_\mathcal{F}$.

1.11. According to Proposition 1.9, when considering the *generator families* of $\mathfrak{w}_\mathcal{F}$, it suffices to consider the $\mathcal{F}$-essential subgroups of P . Now, if Q is an $\mathcal{F}$-essential subgroup of P , we have $\mathfrak{h}_\mathcal{F}(Q)/\mathfrak{r}_\mathcal{F}(Q) \ncong \mathbb{Z}$ and, denoting by $\overline{\mathcal{F}(P,Q)}$ the image of $\mathcal{F}(P,Q)$ in the quotient $\mathfrak{h}_\mathcal{F}(Q)/\mathfrak{r}_\mathcal{F}(Q)$, it is clear that $\mathcal{F}(Q)$ acts on $\overline{\mathcal{F}(P,Q)}$ by composition on the left. At this point, it follows from [5, Theorem 5.11] that:

1.11.1 *If $\mathcal{F}$ is a Frobenius P-category then $\mathcal{F}(Q)$ is transitive on $\overline{\mathcal{F}(P,Q)}$, Q is $\mathcal{F}$-selfcentralizing and we have $\mathbf{O}_p\big(\mathcal{F}(Q)\big) = \mathcal{F}_Q(Q)$.*

Recall that Q is an $\mathcal{F}$-*selfcentralizing* subgroup of P if $C_P\big(\varphi(Q)\big) = Z\big(\varphi(Q)\big)$ for any $\varphi \in \mathcal{F}(P,Q)$ [5, 4.8] and let us say that Q is an $\mathcal{F}$-*radical* if it is

$\mathcal{F}$-selfcentralizing and we have $\mathbf{O}_p\big(\mathcal{F}(Q)\big) = \mathcal{F}_Q(Q)$. Moreover, recall that Q is an $\mathcal{F}$-*intersected* subgroup of P if it is selfcentralizing and fulfills [5, 4.11]

$$\mathcal{F}_Q(Q) = \bigcap_{\varphi\in\mathcal{F}(P,Q)} {}^{\varphi^*}\mathcal{F}_P\big(\varphi(Q)\big) \qquad 1.11.2;$$

actually, an $\mathcal{F}$-radical is an $\mathcal{F}$-intersected subgroup of P . Note that statement 1.11.1, Proposition 1.9 and Lemma 1.5 already prove the corresponding version in $\mathcal{F}$ of Alperin's Fusion Theorem [5, Corollary 5.14]; thus, we consider the following condition on $\mathcal{F}$:

Alperin condition. *For any $\mathcal{F}$-essential subgroup Q of P , Q is an $\mathcal{F}$-radical and $\mathcal{F}(Q)$ acts transitively on $\overline{\mathcal{F}(P,Q)}$.*

1.12. On the other hand, for any subgroup Q of P and any subgroup K of $\mathrm{Aut}(Q)$ such that Q is fully K-normalized in $\mathcal{F}$, recall that the *divisible $N_P^K(Q)$-subcategory $N_\mathcal{F}^K(Q)$ of $\mathcal{F}$* [5, 2.14] is the subcategory of $\mathcal{F}$ where, for any pair of subgroups R and T of $N_P^K(Q)$, the set of morphisms from T to R is the set of elements $\varphi \in \mathcal{F}(R,T)$ fulfilling the following condition:

1.12.1 *There are an $\mathcal{F}$-morphism $\psi : Q{\cdot}T \to Q{\cdot}R$ and an element χ of K such that $\chi(u) = \psi(u)$ for any $u \in Q$ and that $\psi(v) = \varphi(v)$ for any $v \in T$.*

Note that, if $\mathcal{F}$ is a Frobenius P-category then $N_\mathcal{F}^K(Q)$ is a Frobenius $N_\mathcal{F}^K(Q)$-category too [5, Proposition 2.16]. Our main purpose in this Note is to prove the following result.

Theorem 1.13. *A divisible P-category is a Frobenius P-category if and only if, for any subgroup Q of P and any subgroup K of* $\mathrm{Aut}(Q)$ *such that Q is fully K-normalized in $\mathcal{F}$, the $N_P^K(Q)$-category $N_\mathcal{F}^K(Q)$ fulfills the Sylow and the Alperin conditions.*

2. Auxiliary results

2.1. In order to prove Theorem 1.13, it is handy to consider *partial Frobenius P-categories* in the following sense. First of all, for short we say that a triple (Q,K,φ) formed by a subgroup Q of P , a subgroup K of $\mathrm{Aut}(Q)$ and an $\mathcal{F}$-morphism $\varphi : Q \to P$ is *extensile* whenever there are an $\mathcal{F}$-morphism $\psi : Q{\cdot}N_P^K(Q) \to P$ and an element χ of $K \cap \mathcal{F}(Q)$ such that $\psi(u) = \varphi\big(\chi(u)\big)$ for any $u \in Q$; thus, the *extension condition* above states that such a triple (Q,K,φ) which fulfills that $\varphi(Q)$ is fully ${}^\varphi K$-normalized in $\mathcal{F}$ is *extensile*.

2.2. Let $\mathfrak{X}$ be a nonempty set of subgroups of P containing any subgroup Q of P such that $\mathcal{F}(Q,R) \neq \emptyset$ for some $R \in \mathfrak{X}$, and denote by $\mathcal{F}^{^\mathfrak{X}}$ the *full* subcategory of $\mathcal{F}$ over $\mathfrak{X}$; we say that $\mathcal{F}^{^\mathfrak{X}}$ is a *partial Frobenius P-category* if *$\mathcal{F}$ fulfills the Sylow condition and any triple (Q,K,φ) formed by an element Q*

of $\mathfrak{X}$, a subgroup K of $\mathrm{Aut}(Q)$ *and an $\mathcal{F}$-morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully ${}^{\varphi}K$-normalized in $\mathcal{F}$ is extensile.* From the proof of [5, Corollary 2.13] it is straightforward to prove the following criterion that we need here.

Proposition 2.3. *With the notation above, assume that $\mathcal{F}$ fulfills the Sylow condition. Then, $\mathcal{F}^{\mathfrak{X}}$ is a partial Frobenius P-category if and only if it fulfills the following two conditions*

2.3.1 *For any pair of $\mathcal{F}$-isomorphic elements Q and Q' of $\mathfrak{X}$, which are both fully normalized and fully centralized in $\mathcal{F}$, there is an $\mathcal{F}$-isomorphism $N_P(Q) \cong N_P(Q')$ mapping Q onto Q'.*

2.3.2 *For any element Q of $\mathfrak{X}$ fully normalized and fully centralized in $\mathcal{F}$ and any subgroup R of $N_P(Q)$ containing $Q{\cdot}C_P(Q)$, denoting by $\mathcal{F}(R)_Q$ the stabilizer of Q in $\mathcal{F}(R)$, the group homomorphism $\mathcal{F}(R)_Q \to N_{\mathcal{F}(Q)}\big(\mathcal{F}_R(Q)\big)$ induced by the restriction is surjective.*

2.4. Similarly, note that all the definitions in 1.6, 1.7 and 1.8 above can be done in $\mathcal{F}^{\mathfrak{X}}$ and then an element Q of $\mathfrak{X}$ is $\mathcal{F}^{\mathfrak{X}}$-essential if and only if it is $\mathcal{F}$-essential; moreover, if $\mathcal{F}^{\mathfrak{X}}$ is a partial Frobenius P-category, the characterization of the $\mathcal{F}$-essential subgroups Q in [5, Theorem 5.11] still holds in $\mathcal{F}^{\mathfrak{X}}$. Here, we also need the lemma [5, Lemma 4.13] which can be restated as follows.

Lemma 2.5. *With the notation above, assume that $\mathcal{F}^{\mathfrak{X}}$ is a partial Frobenius P-category. Then, a triple (R, J, ψ) formed by a subgroup R of P, a subgroup J of* $\mathrm{Aut}(R)$ *and an $\mathcal{F}$-morphism $\psi : R \to P$ such that $\psi(R)$ is fully ${}^{\psi}J$-normalized in $\mathcal{F}$ is extensile provided there are $Q \in \mathfrak{X}$ having R as a normal subgroup and stabilizing J, and an $\mathcal{F}$-morphism $\eta : Q \to P$ extending ψ.*

2.6. Finally, we need the following characterization of the Frobenius P-categories [5, Theorem 4.12].

Theorem 2.7. *A divisible P-category $\mathcal{F}$ fulfilling the Sylow condition is a Frobenius P-category if and only if the following two conditions hold*

2.7.1 *If Q is an $\mathcal{F}$-intersected subgroup of P, R is a subgroup of $N_P(Q)$ containing Q and $\varphi : Q \to P$ is an $\mathcal{F}$-morphism fulfilling ${}^{\varphi}\mathcal{F}_R(Q) \subset \mathcal{F}_P\big(\varphi(Q)\big)$ then there is an $\mathcal{F}$-morphism $\psi : R \to P$ extending φ.*

2.7.2 *Any divisible P-category $\mathcal{F}'$ fulfilling $\mathcal{F}'(P,Q) \supset \mathcal{F}(P,Q)$ for every $\mathcal{F}$-intersected subgroup Q of P contains $\mathcal{F}$.*

3. Proof of Theorem 1.13.

3.1. If $\mathcal{F}$ is a Frobenius P-category then it follows from [5, Proposition 2.16] that the $N_P^K(Q)$-category $N_{\mathcal{F}}^K(Q)$ above is a Frobenius $N_P^K(Q)$-category and therefore it fulfills the Sylow and the Alperin conditions (cf. 1.10 and 1.11).

3.2. Conversely, assume that for any subgroup Q of P and any subgroup K of $\mathrm{Aut}(Q)$ such that Q is fully K-normalized in $\mathcal{F}$, the $N_P^K(Q)$-category $N_{\mathcal{F}}^K(Q)$ fulfills the Sylow and the Alperin conditions; we argue by induction on $|P|$, $\prod_Q |\mathcal{F}(P,Q)|$ where Q runs over the set of subgroups of P, and $|\mathfrak{X}|$ successively; since $\mathcal{F}$ fulfills the Sylow condition, we may assume that $\mathcal{F}^{\mathfrak{X}}$ is a partial Frobenius P-category but $\mathfrak{X}$ does not coincide with the set of all the subgroups of P.

3.3. Let Q be a maximal subgroup of P which does not belong to $\mathfrak{X}$; setting

$$\mathfrak{Y} = \mathfrak{X} \cup \{\varphi(Q)\}_{\varphi \in \mathcal{F}(P,Q)} \qquad 3.3.1,$$

it suffices to prove that $\mathcal{F}^{\mathfrak{Y}}$ fulfills both conditions in Proposition 2.3 above; actually, we may assume that $Q \neq \{1\}$. Let $\varphi : Q \to P$ be an $\mathcal{F}$-morphism, set $Q' = \varphi(Q)$ and assume that Q and Q' are different and both fully normalized and fully centralized in $\mathcal{F}$; then, according either to the very definition of $\mathcal{F}$-essential subgroup or to the Alperin condition, in both cases the image $\bar{\varphi}$ of φ in $\overline{\mathcal{F}(P,Q)}$ coincides with $\overline{\iota_Q^P \circ \sigma}$ for some $\sigma \in \mathcal{F}(Q)$, where ι_Q^P denotes the inclusion map $Q \to P$; that is to say, the difference $\varphi - \iota_Q^P \circ \sigma$ is $\mathcal{F}$-reducible and therefore we have (cf. 1.8.1)

$$\varphi - \iota_Q^P \circ \sigma = \sum_R \sum_{\theta \in \mathfrak{w}_{\mathcal{F}}(R)} \theta \circ \alpha_{R,\theta} \qquad 3.3.2$$

for suitable $\alpha_{R,\theta} \in \mathbb{Z}\mathcal{F}(R,Q)$, where R runs over the set of subgroups of P such that $|R| > |Q|$.

3.4. Consequently, it follows from 1.3 that we still have

$$\varphi - \iota_Q^P \circ \sigma = \sum_{j \in J} (\psi'_j - \psi_j) \circ \mu_j \qquad 3.4.1$$

where J is a nonempty finite set and where, for any $j \in J$, ψ_j and ψ'_j are elements of $\mathcal{F}(P, R_j)$ and $\mu_j \in \mathcal{F}(R_j, Q)$ for a suitable subgroup R_j of P such that $|R_j| > |Q|$; more precisely, applying again the Alperin condition and arguing by induction on $|P:Q|$, we actually get

$$\varphi - \iota_Q^P \circ \sigma = \sum_{i \in I} \iota_{U_i}^P \circ (\tau_i - \mathrm{id}_{U_i}) \circ \nu_i \qquad 3.4.2$$

where I is a nonempty finite set and, for any $i \in I$, τ_i is an element of $\mathcal{F}(U_i)$ and $\nu_i \in \mathcal{F}(U_i, Q)$ for a suitable subgroup U_i of P such that $|U_i| > |Q|$.

3.5. Then, it follows from Lemma 1.5 that, for a suitable ℓ, we can identify Δ_ℓ with a subset of I in such a way that $Q_0 = Q$, $Q_{i+1} = \tau_i(Q_i)$ for any $i \in \Delta_\ell$, $Q_{\ell+1} = Q'$ and, denoting by $\varphi_i : Q_i \cong Q_{i+1}$ the $\mathcal{F}$-isomorphism induced by τ_i, the composition of all these isomorphisms coincides with the isomorphism $Q \cong Q'$ induced by $\varphi \circ \sigma^{-1}$. Moreover, note that U_i contains Q_i and Q_{i+1} for any $i \in \Delta_\ell$ and, in particular, it belongs to $\mathfrak{X}$.

3.6. For any $i \in \Delta_{\ell+1}$, let us choose $\eta_i \in \mathcal{F}\big(P, N_P(Q_i)\big)$ such that $R_i = \eta_i(Q_i)$ is fully normalized in $\mathcal{F}$ [5, Proposition 2.7] and we may assume that $R_0 = Q$, that $R_{\ell+1} = Q'$ and that η_0 and $\eta_{\ell+1}$ are the corresponding inclusion maps; moreover, for any $i \in \Delta_\ell$, denote by $\psi_i : R_i \cong R_{i+1}$ the $\mathcal{F}$-morphism mapping $\eta_i(u)$ on $\eta_{i+1}\big(\varphi_i(u)\big)$ for any $u \in Q_i$. Then, for any $i \in \Delta_\ell$ we claim that we can apply Lemma 2.5 above to the triple $(R_i, \mathrm{Aut}(R_i), \psi_i)$; indeed, we are assuming that $\mathcal{F}^{\mathfrak{x}}$ is a partial Frobenius P-category; moreover, it is clear that R_i is a proper normal subgroup of $\eta_i\big(N_{U_i}(Q_i)\big)$ which clearly stabilizes $\mathrm{Aut}(R_i)$; finally, the $\mathcal{F}$-morphism

$$\eta_i\big(N_{U_i}(Q_i)\big) \longrightarrow \eta_{i+1}\big(N_{U_i}(Q_{i+1})\big) \qquad 3.6.1$$

mapping $\eta_i(v)$ on $\eta_{i+1}\big(\tau_i(v)\big)$ for any $v \in N_{U_i}(Q_i)$ clearly extends ψ_i.

3.7. Hence, since R_{i+1} is fully normalized in $\mathcal{F}$, it follows from this lemma that there is an $\mathcal{F}$-morphism $\zeta_i : N_P(R_i) \to P$ extending $\chi_i \circ \psi_i$ for some $\chi_i \in \mathcal{F}(R_i)$; moreover, since R_i is fully normalized in $\mathcal{F}$, we actually get

$$\zeta_i\big(N_P(R_i)\big) = N_P(R_{i+1}) \qquad 3.7.1,$$

so that ζ_i induces an $\mathcal{F}$-isomorphism $\xi_i : N_P(R_i) \cong N_P(R_{i+1})$ mapping R_i onto R_{i+1}. Finally, the composition of all these $\mathcal{F}$-isomorphisms when i runs over Δ_ℓ yields an $\mathcal{F}$-isomorphism $N_P(Q) \cong N_P(Q')$ which maps Q onto Q', proving condition 2.3.1.

3.8. In order to prove condition 2.3.2, we set $P' = N_P(Q)$ and we claim that the P'-category $\mathcal{F}' = N_{\mathcal{F}}(Q)$ still fulfills our hypothesis in 3.2 above; more explicitly, if R is a subgroup of P' and J a subgroup of $\mathrm{Aut}(R)$ such that R is fully J-normalized in $\mathcal{F}'$, we claim that the $N_{P'}^J(R)$-category $N_{\mathcal{F}'}^J(R)$ fulfills the Sylow and the Alperin conditions. Set $T = Q{\cdot}R$ and denote by I the subgroup of automorphisms of T which stabilize Q and R, and act on R *via* elements of J; then, from its very definition (cf. 1.12), it is easily checked that

$$N_P^I(T) = N_{P'}^J(R) \quad \text{and} \quad N_{\mathcal{F}}^I(T) = N_{\mathcal{F}'}^J(R) \qquad 3.8.1;$$

hence, in order to prove our claim, it suffices to prove that T is fully I-normalized in $\mathcal{F}$.

3.9. We actually follow the proof of [5, Lemma 2.17]; for any $\mathcal{F}$-morphism $\psi : T{\cdot}N_P^I(T) \to P$, set $Q' = \psi(Q)$, denote by $\psi^* : Q' \cong Q$ the inverse of the isomorphism $Q \cong Q'$ determined by ψ, and consider the $\mathcal{F}$-morphism

$$\iota_Q^P \circ \psi^* : Q' \longrightarrow P \qquad 3.9.1$$

where $\iota_Q^P : Q \to P$ is the inclusion map; it follows from [5, Proposition 2.7] that there is $\xi : N_P(Q') \to P$ such that $Q'' = \xi(Q')$ is both fully centralized

and fully normalized in $\mathcal{F}$, and therefore, since Q is both fully centralized and fully normalized in $\mathcal{F}$, it follows from our argument above applied to Q'' and to Q that there is an $\mathcal{F}$-morphism

$$\zeta : N_P(Q') \longrightarrow P \qquad 3.9.2$$

mapping Q' onto Q. In particular, we have $\zeta\big(N_P(Q')\big) \subset P'$ and, since $\psi\big(T{\cdot}N_P^I(T)\big)$ normalizes Q', the homomorphism

$$\eta : T{\cdot}N_P^I(T) = Q{\cdot}\big(R{\cdot}N_{P'}^J(R)\big) \longrightarrow P' \qquad 3.9.3$$

mapping $w \in T{\cdot}N_P^I(T)$ on $\zeta\big(\psi(w)\big)$ belongs to $\mathcal{F}\big(P', T{\cdot}N_P^I(T)\big)$; moreover, since $R{\cdot}N_{P'}^J(R) \subset T{\cdot}N_P^I(T)$ and $\zeta\big(\psi(Q)\big) = Q$, η determines an $\mathcal{F}'$-morphism from $N_{P'}^J(R)$ to P' (cf. 1.12.1); hence, since R is fully J-normalized in $\mathcal{F}'$, we get (cf. 3.8.1)

$$\zeta\Big(\psi\big(N_P^I(T)\big)\Big) = \eta\big(N_{P'}^J(R)\big) = N_{P'}^{\eta J}\big(\eta(R)\big) \supset \zeta\Big(N_P^{\psi I}\big(\psi(T)\big)\Big) \qquad 3.9.4$$

which forces $\psi\big(N_P^I(T)\big) = N_P^{\psi I}\big(\psi(T)\big)$, proving the claim.

3.10. Consequently, if $\mathcal{F}' \neq \mathcal{F}$ then it follows from the induction hypothesis that $\mathcal{F}'$ is a Frobenius P'-category and, in particular, it fulfllls the corresponding condition 2.3.2; thus, since Q is still fully normalized and fully centralized in $\mathcal{F}'$, for any subgroup R of $P' = N_P(Q)$ containing $Q{\cdot}C_{P'}(Q)$, the restriction induces a surjective group homomorphism

$$\mathcal{F}(R)_Q = \mathcal{F}'(R)_Q \longrightarrow N_{\mathcal{F}'(Q)}\big(\mathcal{F}_R(Q)\big) = N_{\mathcal{F}(Q)}\big(\mathcal{F}_R(Q)\big) \qquad 3.10.1,$$

so that, in this case, $\mathcal{F}$ also fulfills condition 2.3.2.

3.11. Finally, assume that $P' = P$ and $\mathcal{F}' = \mathcal{F}$; we claim that any $\mathcal{F}$*-intersected* subgroup R of P (cf. 1.11) contains Q; indeed, since we have

$$\mathcal{F}(P,R) = \big(N_{\mathcal{F}}(Q)\big)(P,R) \qquad 3.11.1,$$

any $\psi \in \mathcal{F}(P,R)$ can be extended to some $\hat\psi \in \mathcal{F}(P, Q{\cdot}R)$ and therefore we have

$$\hat\psi\big(N_Q(R)\big) \subset N_P\big(\psi(R)\big) \qquad 3.11.1,$$

so that we get $\mathcal{F}_Q(R) \subset {}^{\psi^*}\mathcal{F}_P\big(\psi(R)\big)$; thus, according to equality 1.11.2, we still have $\mathcal{F}_Q(R) \subset \mathcal{F}_R(R)$ and therefore $N_Q(R) \subset R$, which forces $Q \subset R$.

3.12. Firstly assume that Q is *not* $\mathcal{F}$*-intersected*; then, we claim that $\mathcal{F}$ fulfills the two conditions in Theorem 2.7 above, so that $\mathcal{F}$ is a Frobenius P-category and, in particular, it fulfills condition 2.3.2. According to 3.11 and to our choice of Q, any $\mathcal{F}$-intersected subgroup of P belongs to $\mathfrak{X}$ and therefore, since we are assuming that $\mathcal{F}^{\mathfrak{X}}$ is a partial Frobenius P-category, condition 2.7.1 holds.

3.13. Moreover, since any $\mathcal{F}$-essential subgroup U of P is $\mathcal{F}$-intersected (cf. 1.11), U belongs to $\mathfrak{X}$ and we claim that any divisible P-category $\hat{\mathcal{F}}$ fulfilling $\hat{\mathcal{F}}(P,U) \supset \mathcal{F}(P,U)$ for every $\mathcal{F}$-essential subgroup U of P contains $\mathcal{F}$; indeed, let R be a subgroup of P and $\psi : R \to P$ an $\mathcal{F}$-morphism; we may assume that R is not $\mathcal{F}$-essential and then, as in 3.4 above, it follows from Lemma 1.5, Proposition 1.9 and the Alperin condition that we have

$$\begin{aligned} \iota_R^P &= \iota_{U_0}^P \circ \nu_0 \\ \iota_{U_{i-1}}^P \circ \sigma_{i-1} \circ \nu_{i-1} &= \iota_{U_i}^P \circ \nu_i \quad \text{for any } 1 \le i \le \ell \\ \iota_{U_\ell}^P \circ \sigma_\ell \circ \nu_\ell &= \psi\,. \end{aligned} \qquad 3.13.1$$

for some ℓ and, for any $i \in \Delta_\ell$, a suitable $\mathcal{F}$-essential subgroup U_i of P and some elements $\sigma_i \in \mathcal{F}(U_i)$ and $\nu_i \in \mathcal{F}(U_i, R)$; then, since $\hat{\mathcal{F}}$ is divisible, we have $\nu_0 = \iota_R^{U_0}$ and, in particular, it belongs to $\hat{\mathcal{F}}(U_0, R)$; arguing by induction on ℓ, we may assume that $\nu_{\ell-1} \in \hat{\mathcal{F}}(U_{\ell-1}, R)$ and, since $\sigma_{\ell-1}$ belongs to $\mathcal{F}(U_{\ell-1}) \subset \hat{\mathcal{F}}(U_{\ell-1})$, we get $\nu_\ell \in \hat{\mathcal{F}}(U_\ell, R)$, so that ψ belongs to $\hat{\mathcal{F}}(P,R)$ since $\sigma_\ell \in \mathcal{F}(U_\ell) \subset \hat{\mathcal{F}}(U_\ell)$.

3.14. Secondly, assume that Q is $\mathcal{F}$*-intersected*; since we are assuming that $N_{\mathcal{F}}(Q) = \mathcal{F}$, it is easily checked that, in this case, equality 1.11.2 forces $\mathbf{O}_p\big(\mathcal{F}(Q)\big) = \mathcal{F}_Q(Q)$; moreover, since Q is $\mathcal{F}$-selfcentralizing, in order to prove that condition 2.3.2 holds it suffices to consider a subgroup R of P strictly containing Q and then we have $\mathcal{F}_R(Q) \neq \mathcal{F}_Q(Q)$, so that the normalizer $K = N_{\mathcal{F}(Q)}\big(\mathcal{F}_R(Q)\big)$ is a proper subgroup of $\mathcal{F}(Q)$.

3.15. At present, set $P'' = N_P^K(Q)$ and $\mathcal{F}'' = N_{\mathcal{F}}^K(Q)$; note that P'' contains R, that $\mathcal{F}''$ fulfills the Sylow and the Alperin conditions (cf. 3.2), and that we have $\mathcal{F}''(Q) = K$; since we also have

$$\mathcal{F}_Q(Q) \neq \mathcal{F}_R(Q) \lhd \mathcal{F}''(Q) = K \qquad 3.15.1,$$

Q is *not* $\mathcal{F}''$-essential; thus, any nonidentity element $\varphi \in \mathcal{F}''(Q)$ defines a $\mathcal{F}''$*-reducible* $\mathcal{F}''$*-dimorphism* $\iota_Q^{P''} \circ (\varphi - \mathrm{id}_Q)$ and therefore, as in 3.4 above, it follows from Lemma 1.5, Proposition 1.9 and the Alperin condition that we have

$$\begin{aligned} \iota_Q^{P''} &= \iota_{U_0}^{P''} \circ \nu_0 \\ \iota_{U_{i-1}}^{P''} \circ \sigma_{i-1} \circ \nu_{i-1} &= \iota_{U_i}^{P''} \circ \nu_i \quad \text{for any } 1 \le i \le \ell \\ \iota_{U_\ell}^{P''} \circ \sigma_\ell \circ \nu_\ell &= \iota_Q^{P''} \circ \varphi\,. \end{aligned} \qquad 3.15.2$$

for some ℓ and, for any $i \in \Delta_\ell$, for a suitable $\mathcal{F}''$-essential subgroup U_i of P'' and some elements $\sigma_i \in \mathcal{F}''(U_i)$ and $\nu_i \in \mathcal{F}''(U_i, Q)$; note that, since we have $U_i \subset P''$, the image of U_i in $\mathcal{F}(Q)$ normalizes $\mathcal{F}_R(Q)$ and therefore, since Q is $\mathcal{F}$-selfcentralizing, U_i normalizes R.

3.16. Then, for any $i \in \Delta_\ell$, we claim that we can apply Lemma 2.5 to $\mathcal{F}^x$ and to the triple $(Q, \mathcal{F}_R(Q), \varphi_i)$ where $\varphi_i : Q \to P$ is the $\mathcal{F}$-morphism defined by the restriction of σ_i; indeed, Q is a normal proper subgroup of U_i, U_i stabilizes $\mathcal{F}_R(Q)$ and the $\mathcal{F}$-morphism $\iota_{U_i}^P \circ \sigma_i : U_i \to P$ extends φ_i. Consequently, it follows from this lemma that this triple is *extensile* and therefore, since $N_P^{\mathcal{F}_R(Q)}(Q) = R$, there exists an $\mathcal{F}$-morphism $\psi_i : R \to P$ extending φ_i; moreover, since φ_i is the restricion of $\sigma_i \in \mathcal{F}''(U_i)$, φ_i normalizes $\mathcal{F}_R(Q)$ and therefore, since Q is $\mathcal{F}$-selfcentralizing, we get $\psi_i(R) = R$. Finally, the composition of the family of $\mathcal{F}$-automorphisms of R determined by $\{\psi_i\}_{i \in \Delta_\ell}$ coincides with φ; that is to say, the group homomorphism

$$\mathcal{F}(R) \longrightarrow N_{\mathcal{F}(Q)}\big(\mathcal{F}_R(Q)\big) \qquad 3.16.1$$

induced by the restriction is surjective, proving condition 2.3.2. We are done.

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